Problem 1. Potential games (5 points)

a) Consider the two-player potential game, where the row player with actions: U, D (for up and down rows) aims to minimize the cost function $A \in \mathbb{R}^{2\times 2}$ and the column player with actions: L, R (for left and right columns) aims to minimize the cost $B \in \mathbb{R}^{2\times 2}$, with $A = B = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$.

i) What is(are) the potential function minimizer(s)? (.5 point)
Solution: (U, L)

ii) What is(are) the Nash equilibrium(equilibria) of the game? (2 points)
Solution: $(U, L), (D, R)$

iii) Is there an admissible Nash equilibrium? If so, provide it. Else, justify. (.5 point)

Solution: Yes, (U, L)

b) Consider a zero-sum game with the cost function of the minimizer (row player) being $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that the game has an exact potential function if and only if $a + d = b + c$. (2 points)

Solution: Write the four linear equations, two for each player, on what the potential function should satisfy:

By definition of an exact potential function and the fact that the game is zero-sum, the function has to satisfy:

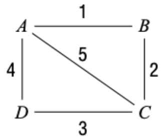
$$\begin{aligned} a - c &= P_{11} - P_{21} \\ b - d &= P_{12} - P_{22} \\ -a + b &= P_{11} - P_{12} \\ -c + d &= P_{21} - P_{22} \end{aligned}$$

Now, add the first equation to the fourth, and subtract from this sum the second and third equation to get the result.

Problem 2. Best-response and congestion games (6 points)

Consider the following road network that connects the four cities A, B, C, D through five roads. There are 10 vehicles and they have two types. Type 1: 6 vehicles with origin A and destination D; Type 2: 4 vehicles with origin B and destination D. Let $C(n) > 0$ denote the cost of traveling road $r \in \{1, 2, 3, 4, 5\}$, where n is the number of vehicles using a given road and $C : \mathbb{N} \to \mathbb{R}$ is monotonically increasing. So, the action space for Type 1 players is $\Gamma := \{\{4\}, \{5, 3\}, \{1, 2, 3\}\}$.

a) What is the action space, Σ, for Type 2 players? (1 point)
Solution: $\{2, 3\}, \{1, 4\}, \{1, 5, 3\}, \{2, 5, 4\}$



From now on, consider the initial strategies where vehicles in each type distribute equally to their respective feasible action spaces. So, from Type 1 vehicles, 2 choose each of the three strategies in Γ.

b) What is the cost of a player choosing $\{1, 2, 3\}$? (2 points)

Solution: If we write the number of players using each of the seven strategies, we find $1 : 4$, $2 : 4$, $3 : 6$, $4 : 4$, and $5 : 4$, where the notation $r : n$ denotes the road r and n the number of players using r. So, this players' cost is $C(4) + C(4) + C(6)$.

c) Consider a player who chose $\{1, 2, 3\}$. What is this player's best response strategy? (1 point)

Solution: She can switch to strategy $\{4\}$ for a new cost of $C(5)$.

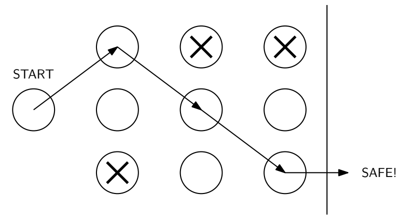
d) Justify that this game has a Nash equilibrium strategy. (1 point)

Solution: It is a congestion game, and as we saw in lecture 3, this is a potential game.

e) What is an upper bound on the number of iterations for which the best-response dynamic in the above game will converge to a Nash equilibrium? (1 point)

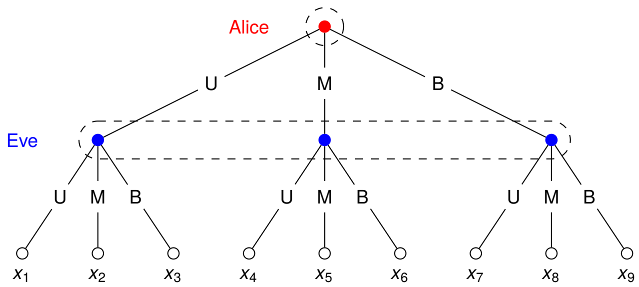
Solution: The domain of the potential function is $\Gamma \times \Sigma$, which has cardinality: $3^6 \times 4^4$. Since the best response results in an improvement in every strategy, in at most the above number of iterations, a Nash equilibrium is reached. Note that this is a very crude upper bound.

Problem 3. Multi-stage game (9 points)



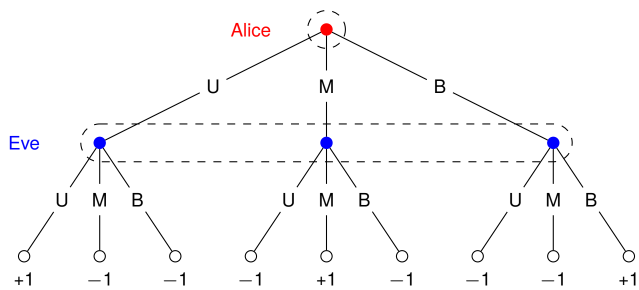
Player 1 (Alice) is trying to escape, going from the **start node** to the **safe zone** without being intercepted. At every stage of the game, Alice moves one step closer to the safe zone. She can decide to continue on the same row, or instead move diagonally one row up or one row down. Player 2 (Eve) is trying to stop Alice. At each stage, Eve is aware of Alice's current position, and she is allowed to block one of the three rows in the next stage, taking her decision simultaneously with Alice. If she selects the row corresponding to Alice's next move, she stops her and she wins the game getting a cost of -1, while Alice gets a cost of 1.

Let U, M, B denote the choice of Up, Middle and Bottom rows, respectively. First, consider the case with 3 rows and only 1 stage (that is, only 1 chance for Eve to stop Alice). The game tree for Alice is shown here.



a) Fill in the x_i's above. Formulate the game in matrix form. (1 point)

Solution: The game tree is shown below. U represents the player choosing the top row, M represents the player choosing the middle row, and B represents the player choosing the bottom row. The costs for each leaf are those of Alice. Note that the game is zero-sum, so Eve's costs are the costs of Alice multiplied by -1.



The game's matrix with Alice as the row player and Eve as the column player is:

$$\begin{array}{c} \\ U \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} U & M & B \end{array} \\ \begin{bmatrix} (+1,-1) & (-1,+1) & (-1,+1) \\ (-1,+1) & (+1,-1) & (-1,+1) \\ (-1,+1) & (-1,+1) & (+1,-1) \end{bmatrix} \end{array}$$

b) Show that $y^* = z^* = (1/3, 1/3, 1/3)$ is the Nash equilibrium of the game. (1 point)

Solution: For Alice, we have that:

$$(y^*)^\top B = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 1/3 * \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

and for Eve, we have that:

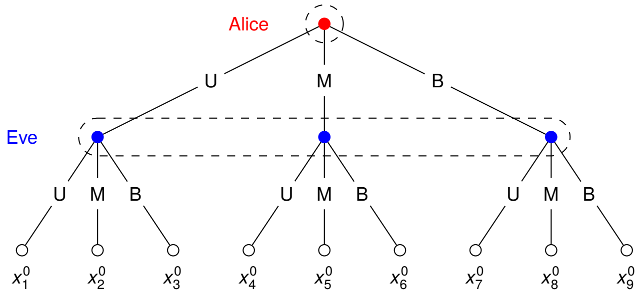
$$Az^* = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = -1/3 * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, $y^* = z^* = (1/3, 1/3, 1/3)$ is the Nash equilibrium of the game.

c) Verify that the value of the game is $-1/3$. (1 point)

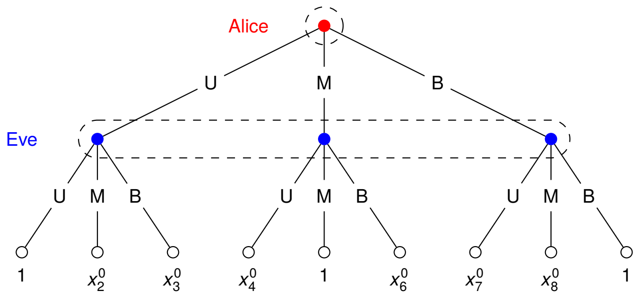
Solution: The value of the game is $J(y^*, z^*) = [1/3, 1/3, 1/3] \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = -1/3$

Now, consider the case where there are 3 rows and 2 stages (that is, 2 chances for Eve to stop Alice). We aim to use backward induction to determine the value of the game and fill in the entries denoted as x_i's below, corresponding to Alice's costs.



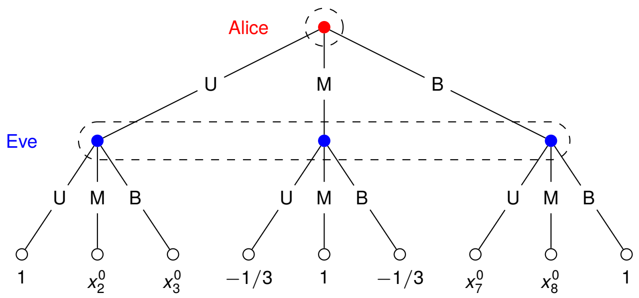
d) First, if Alice and Eve choose the same move in the first stage, clearly Eve catches Alice and the game ends. Based on this, fill in x_1^0, x_5^0, x_9^0 (.5 point).

Solution:



e) Next, note that if Alice chooses M in the first stage and Eve does not choose M, the game at the next stage will be identical to the one discussed in the previous part. Based on this observation, fill in x_4^0, x_6^0. (1 point)

Solution: If Alice had chosen to move to the middle row initially and Eve had not chosen M, then Alice would be able to pick any of $\{U, M, B\}$. Thus, if Alice and Eve had played (M, U) or (M, B), then the games following information set $\mathcal{I}_{M,U}$ and $\mathcal{I}_{M,B}$ correspond to the case studied earlier, with 3 rows and 1 stage. Thus, the optimal strategy for Alice at these information sets is $y^* = (1/3, 1/3, 1/3)$ and the values of the games are $-1/3$.



f) Now, consider Alice choosing U and Eve choosing M or B, in the first stage. In the next stage, Alice can choose either U or M. Draw the matrix game for this next stage (1 point); Verify that $y^* = [1/2, 1/2]$, and $z^* = [1/2, 1/2, 0]$ is a Nash equilibrium. Based on this fill in x_2^0, x_3^0. (1 point)

Solution: The game's matrix with Alice as the row player and Eve as the column player is:

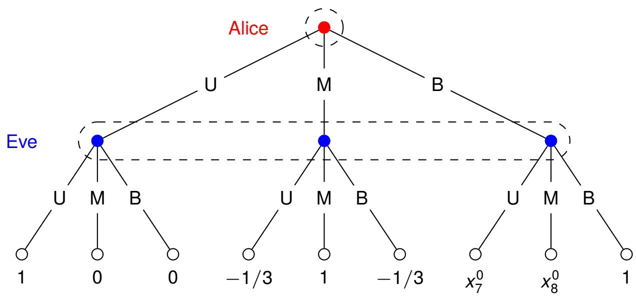
$$\begin{array}{c} \\ U \\ M \end{array} \begin{array}{c} \begin{array}{ccc} U & M & B \end{array} \\ \begin{bmatrix} (+1,-1) & (-1,+1) & (-1,+1) \\ (-1,+1) & (+1,-1) & (-1,+1) \end{bmatrix} \end{array}$$

One can verify that $y^* = [1/2, 1/2]$, and $z^* = [1/2, 1/2, 0]$ is a Nash equilibrium by showing that the following equations hold

$$(y^*)^\top Az^* \leq y^\top Az^*, \quad \forall \text{pure strategy } y$$
$$(y^*)^\top - Az^* \leq (y^*)^\top Az, \quad \forall \text{pure strategy } z$$

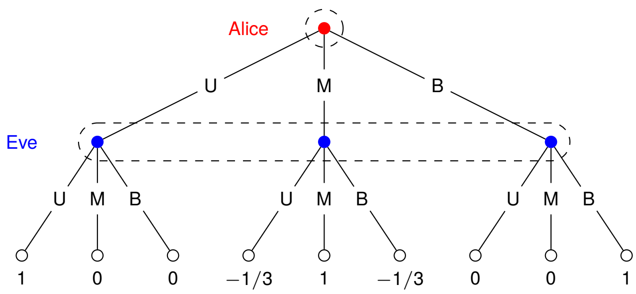
with $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}$.

The value of the game is $J(y^*, z^*) = [1/2, 1/2] \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = 0$. The tree is given by:



g) By symmetry, fill in x_6^0, x_7^0. (.5 point)

Solution:



h) Formulate the game with the game tree shown above in matrix form (1 point).

Solution: The game's matrix with Alice as the row player and Eve as the column player is:

$$\begin{array}{c} \\ U \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \quad U \quad & \quad M \quad & \quad B \quad \end{array} \\ \begin{bmatrix} (+1,-1) & (0,0) & (0,0) \\ (-1/3,+1/3) & (+1,-1) & (-1/3,+1/3) \\ (0,0) & (0,0) & (+1,-1) \end{bmatrix} \end{array}$$

i) Verify that the Nash equilibrium strategy of Alice in her first move is $y^* = [4/11, 3/11, 4/11]$. (1 point)

Solution: For Alice, we verify that:

$$(y^*)^\top B = \begin{bmatrix} 4/11 & 3/11 & 4/11 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1/3 & -1 & 1/3 \\ 0 & 0 & -1 \end{bmatrix} = -3/11 \begin{bmatrix} 1 & 1 & 1. \end{bmatrix}$$

To find the Nash equilibrium strategy of Eve we solve:

$$Ax^* = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = q \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$(x^*)^\top 1 = 1.$$

Solving these equations results in the following Nash strategy of Eve: $(x_1^*, x_2^*, x_3^*) = (\frac{3}{11}, \frac{5}{11}, \frac{3}{11})$.